

SERRE MULTIPLICITY QUESTION, MUKAI PAIRING AND HODGE THEORY

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ABSTRACT. The Serre conjecture on positivity of intersection multiplicity in proper intersections over general regular rings, is still a challenging open question. In this article we pretend to apply some general techniques from geometry to this question. Specifically a discussion of relations with Hodge theory, Mukai pairing, and Higher residue is proposed to apply to positivity question.

INTRODUCTION

In 1950's J. Serre generalized the definition of Intersection multiplicity (due to Samuel) of two finitely generated modules over a regular local ring A as an Euler characteristic, namely Tor-formula. He proved the non-negativity of this Euler characteristic in several special cases that were enough for the purpose of geometers, for instance in the case where the ring A is essentially of finite type over a field or a discrete valuation ring k . Serre conjectured the vanishing of multiplicity in non-proper intersections, and its positivity in the proper case, for general regular local rings, [S]. The vanishing part was proved by H. Gillet and C. Soule, [GS1], and also independently by P. Roberts, [RO1]. A proof of non-negativity was given by O. Gabber [RO2]. However, the positivity remained open. I try to investigate some relations between the positivity and Hodge theory, using Fourier-Mukai pairing of categories and Hochschild homology. Serre intersection multiplicity definition, can be lifted to a product on $K_0(A)$. That the K -theory of the regular ring A has an intersection theory. In this way the intersection multiplicity can be written as a cup product in $K_0(A)$. The chern character ch and the Riemann-Roch map transform this product into the Chow ring of A , where one may use a dimension argument in order to establish the vanishing of multiplicity.

One step further is to try to push the above product into cohomology of ambient space by integral transform. This allows some more flexible theory to discuss about positivity. However, it still does not provide a complete answer, because one needs a type of positivity of Poincare product in some Weil cohomology theory of a general regular scheme. In this article we investigate this connection in characteristic 0,

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using de Rham cohomology and its cup product. In characteristic $p > 0$, over a field, we make a discussion related to Grothendieck Standard Conjectures and also higher residue pairing.

The Mukai Pairing generally is a non-degenerate pairing on $HH_*(X)$. However it may also be formulated at the level of de Rham cohomologies. Mukai transform is what that makes the Riemann-Roch map a homomorphism. It is the tool we have used to express the Serre (Cartan-Eilenberg) Euler characteristic as a Hodge theory product. To do this one needs to modify Mukai vector by another class namely Gamma class. The Gamma class appears in the context of Mirror symmetry as a perturbative correction term. The original definition of Mukai vector reflects the Calabi-Yau case, where $\hat{\Gamma} = 0$.

In this text we deal with some inter-relations between multiplicity in algebraic geometry and Hodge theory. We compare the two positivity insights in the two theories. Although these tools need to be more developed but we think it opens some windows toward the open questions involved. This paper lies on a way of inter-relation between Multiplicity in algebraic geometry, Mukai pairing in Mirror symmetry, and Conjectures in Hodge theory. We propose to discuss positivity in these 3 areas by one for another.

1. SERRE MULTIPLICITY CONJECTURE

Let A be a regular local ring, and M, N finitely generated A -modules such that $M \otimes_A N$ has finite length. J. P. Serre [S], defines the intersection multiplicity as

$$(1) \quad \chi^A(M, N) := \sum (-1)^i l(\text{Tor}_i^A(M, N))$$

He proves the basic fact that in this case $\dim M + \dim N \leq \dim A$, will hold and makes the following question, known as Serre Multiplicity conjecture.

- (1) If $\dim M + \dim N < \dim A$, then $\chi^A(M, N) = 0$
- (2) In case $\dim M + \dim N = \dim A$, called proper intersection, $\chi^A(M, N) > 0$.

The vanishing part of the conjecture was proved by P. Roberts and also independently by H. Gillet-C. Soule. The positivity in general is still open.

The condition, M, N both have finite projective dimensions implies that the sums in (1) have finitely many terms. Also the condition, $M \otimes_A N$ has finite length, implies, all the $\text{Tor}_i^A(M, N)$ and hence all $\text{Ext}_A^i(M, N)$ have finite length. This makes the former criteria meaningful. One may explain the Euler characteristic in terms of projective resolutions. By a perfect A -complex we mean a bounded complex of finitely generated free (Projective) A -modules. The support of such complex

would be the closed subspace $\text{Supp}(G_\bullet)$ where the localization $(G_\bullet)_\mathfrak{p}$ has non-trivial homology. Then the dimension of the complex is defined to be $\dim \text{Supp}(G_\bullet)$.

If E_\bullet and F_\bullet be free resolutions of the A -modules M, N (which may be taken to be finite, by the regularity of A), then

$$(2) \quad \chi^A(M, N) = \chi(E_\bullet \otimes F_\bullet) = (-1)^{\text{codim } M} \chi(E_\bullet^* \otimes F_\bullet)$$

where the right hand side is the usual Euler characteristic of the complex $E_\bullet \otimes F_\bullet$. The latter makes sense for the complex is supported on the maximal ideal of A .

2. MUKAI PAIRING AND HOCHSCHILD HOMOLOGY

Let k be a commutative ring. The Hochschild homology and cohomology of an k -algebra A are defined by, $HH_k(A) := \text{Tor}_k^{A^e}(A, A)$, $HH^k(A) := \text{Ext}_{A^e}^k(A, A)$, where $A^e = A^{op} \otimes_k A$. Note that in case A is commutative $A^{op} = A$. The Hochschild homology and cohomology of a regular scheme is a sheafification of this definition using the structure sheaves \mathcal{O}_X . The Denis trace map

$$(3) \quad \text{Den} : K_0(X) \rightarrow HH_0(X)$$

would play the role of classical chern character, whose composition with Hochschild-Konstant-Rosenberg homomorphism induces the usual chern character,

$$(4) \quad K_0(X) \xrightarrow{\text{ch}} HH_0(X) \xrightarrow{HKR} \bigoplus_i H^i(X, \Omega_X^i)$$

The Denis trace can be defined in more general context, even for non-smooth X by

$$\text{ch}([e]) = \text{Tr}(\hat{e})$$

in terms of the idempotent $e \in M_n(A)$, where

$$\hat{e} = e + \sum_{n \geq 1} \frac{(2n)!}{(n!)^2} \left(e - \frac{1}{2}\right) (de)^{2n}$$

As an element of the completion, $\hat{\Omega}(A) = \prod \Omega^i(A)$, [L], [CST].

We only consider regular rings A and regular schemes. Therefore, we have $HH_\bullet(A) = HH_\bullet(\text{perf}(A))$. Thus we identify the theory of Chern characters for K -theory of sheaves or their perfect complexes. The H-K-R would be the isomorphism induced by the map;

$$(5) \quad b_0 \otimes b_1 \otimes \dots \otimes b_r \rightarrow \frac{1}{r!} \cdot b_0 \cdot db_1 \wedge \dots \wedge db_r$$

when X is smooth, [RA1]. The philosophy of Mukai-pairing is to modify ch , by cup product with a class $\sqrt{\text{td}_X}$ such that we obtain a homomorphism in Riemann-Roch theorem. We have

$$\text{td}_X^{1/2} \in \bigoplus_i H^i(X, \Omega_X^i).$$

Let X, Y be complex manifolds, and let $\mathcal{E} \in D^b(X \times Y)$. Let π_X, π_Y be the projections. Define the integral transform with kernel \mathcal{E} by;

$$(6) \quad \Phi_{X \rightarrow Y}^{\mathcal{E}} : D^b(X) \rightarrow D^b(Y), \quad \Phi_{X \rightarrow Y}^{\mathcal{E}}(\cdot) = \pi_{Y,*}(\pi_X^*(\cdot) \otimes \mathcal{E})$$

Similarly for $\mu \in H^*(X \times Y, \mathbb{Q})$

$$(7) \quad \Phi_{X \rightarrow Y}^{\mu} : H^*(X, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q}), \quad \Phi_{X \rightarrow Y}^{\mu}(\cdot) = \pi_{Y,*}(\pi_X^*(\cdot) \otimes \mu)$$

called the integral transform in cohomology associated to μ . The association between objects of $D^b(X \times Y)$ or $H^*(X \times Y)$ is functorial. In order to relate the above two functors we use chern character and Riemann-Roch theorem. The Riemann-Roch theorem states that, if $\pi : X \rightarrow Y$ is a local complete intersection morphism;

$$(8) \quad \pi_*(\text{ch}(\bullet) \text{td}(X)) = \text{ch}(\pi_*(\bullet)) \cdot \text{td}(Y)$$

This suggest to define the Mukai vector of \mathcal{E} as follows,

$$(9) \quad v : D^b(X) \rightarrow H^*(X, \mathbb{Q}), \quad v(\cdot) = \text{ch}(\cdot) \cdot \sqrt{\text{td}(X)}$$

Then the commutativity of the following diagram is straight-forward;

$$(10) \quad \begin{array}{ccc} D^b(X) & \xrightarrow{\Phi_{X \rightarrow Y}^{\mathcal{E}}} & D^b(Y) \\ v \downarrow & & \downarrow v \\ H^*(X, \mathbb{Q}) & \xrightarrow{\phi_{X \rightarrow Y}^{v(\mathcal{E})}} & H^*(Y, \mathbb{C}) \end{array}$$

We will denote $\Phi_* = \Phi_{X \rightarrow Y}^{v(\mathcal{E})}$, where $\Phi = \Phi_{X \rightarrow Y}^{\mathcal{E}}$, and it satisfies the associativity and functorial properties naturally. In case $\Phi = \Phi_{X \rightarrow Y}^{\mathcal{E}}$ be an equivalence of categories, $\Phi_* = \Phi_{X \rightarrow Y}^{v(\mathcal{E})}$ would be an isomorphism, [CA1], [CA2], [CA3].

When X is a projective smooth manifold, the map Φ_* does respects the columns of Hodge diamond;

$$\Phi_* = \phi_{X \rightarrow Y}^{v(\mathcal{E})} : \bigoplus_{p-q=i} H^{p,q}(X) \rightarrow \bigoplus_{p-q=i} H^{p,q}(X)$$

This is for, the class $v(\mathcal{E})$ is a Hodge class. Lets define $\tau : H^*(X, \mathbb{C}) \rightarrow H^*(X, \mathbb{C})$ by

$$(11) \quad \tau(v_0, v_1, \dots, v_{2n}) = (v_0, iv_1, -v_2, \dots, i^{2n}v_{2n})$$

and set,

$$(12) \quad .^\vee : H^*(X, \mathbb{C}) \rightarrow H^*(X, \mathbb{C}), \quad v^\vee = \tau(v) \cdot \frac{1}{\sqrt{ch(\omega_X)}}$$

For $td(T_X^\vee) = td(T_X) \cdot \exp(-c_1(T_X)) = td(T_X) \cdot ch(\omega_X)$.

This operator can also be defined more generally on Hochschild homology. If X is proper and smooth, There is a natural isomorphism $HH_\bullet(X) \cong HH_\bullet(\text{perf}(X))$. When Y is of the same type, an object $\Phi \in \text{perf}(X \otimes Y)$, may be considered as the kernel of an integral transform $\text{perf}(X) \rightarrow \text{perf}(Y)$. Then we would have the induced map

$$\Phi_* : HH_\bullet(X) \rightarrow HH_\bullet(Y)$$

Using Kunneth quasi-isomorphism ,we get a pairing

$$HH_\bullet(\text{perf}(X)) \otimes HH_\bullet(\text{perf}(X)) \rightarrow HH_\bullet(\text{perf}(X \times X)) \rightarrow HH_\bullet(\text{perf}(\mathbb{C})) = \mathbb{C}$$

Which is given by shuffle products, [RA1]. A non-trivial fact is that the above induced map Φ_* will become equivalent to the integral transform induced by Φ .

The Mukai Pairing can be generalized to Hochschild homology as

$$\langle ., . \rangle_M : HH_\bullet(X) \otimes HH_\bullet(X) \rightarrow \mathbb{C}$$

called generalized Mukai pairing. This generalization can be easily written using the isomorphism

$$D : \text{RHom}(\Delta_! \mathcal{O}_X, \Delta_* \mathcal{O}_X) \cong \text{RHom}(\Delta_* \mathcal{O}_X, \Delta_* \omega_X)$$

where $\Delta_! \mathcal{O}_X \cong \Delta_* \omega_X^{-1}$ and ω_X is the dualizing sheaf. Then, the Mukai pairing is

$$v \otimes w \rightarrow \text{tr}_{X \times X}(D(v) \circ w)$$

where tr is Serre duality trace. If

$$.\vee : HH_\bullet(X) \rightarrow HH_\bullet(X)$$

is the involution induced through H-K-R isomorphism by the similar one to be $(-1)^p$ on $H^q(X, \Omega_p)$, as defined before. Then we would have

Theorem 2.1. [RA1] *Suppose X is smooth, then*

$$(13) \quad \langle b^\vee, a \rangle_M = \langle a, b \rangle, \quad a, b \in HH_\bullet(X)$$

Moreover, the generalized Mukai pairing on the Hochschild homology of X satisfies

$$\langle a, b \rangle_M = \int_X I(a)^\vee I(b).td_X, \quad a, b \in HH_\bullet(X)$$

where I is the H-K-R isomorphism.

The Euler pairing on $K_0(X)$ is defined by

$$\chi(\mathcal{E}, \mathcal{F}) := \sum_i (-1)^i \dim \text{Ext}_X^i(\mathcal{E}, \mathcal{F})$$

Assume $H^*(X)$ is equipped with the pairing

$$\langle x, y \rangle := (x \cup y \cup td_X) \cap [X]$$

Then the Riemann-Roch theorem states that, the chern character $ch : K_0 \rightarrow H^*(X)$ is map of inner product spaces. The same fact is true for Hochschild homology and Denis trace map, where we have the compatibility of the two chern character by H-K-R homomorphism, [CA1]. The chern character $ch : K_0 \rightarrow HH_0(X)$ is a map of inner product spaces, in other words for $\mathcal{E}, \mathcal{F} \in D(X)$, we have

$$\langle ch(E), ch(F) \rangle_M = \chi(\mathcal{E}, \mathcal{F})$$

A modification of Mukai pairing is to use the $\hat{\Gamma}_X$ -class instead of the square $\sqrt{td_X}$. Then we replace the Mukai vector by the vector

$$E \mapsto (2\pi i)^{\deg(\cdot)/2} \frac{1}{(2\pi)^{d/2}} \Gamma(X) \wedge ch(E)$$

The cohomology class $\hat{\Gamma}_X$ is defined via the identity $\frac{z}{1-e^{-z}} = e^{i\pi z} \Gamma(1-x) \Gamma(1+x)$ used to share the two factors of $\sqrt{td_X}$ with the other chern classes in the Mukai pairing. It explicitly is given by the formula,

$$\hat{\Gamma}_X = \exp(C.ch_1(T_X) + \sum_{n \geq 2} \frac{\zeta(n)}{n} ch_n(T_X))$$

where $C = \lim_{n \rightarrow \infty} (1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln(n))$ is the Euler constant, ζ is the Riemann zeta. Let's write,

$$H^*(X, \mathbb{Q}) \xrightarrow{\mathfrak{d}} H^*(X, \mathbb{C}) \xrightarrow{\hat{\Gamma}_X \wedge (\bullet)} H^*(X, \mathbb{C}), \quad \mathfrak{d} := (2\pi i)^{\deg(\cdot)/2} \frac{1}{(2\pi)^{d/2}}$$

Previously, we defined the Mukai vector as $\nu(\mathcal{E}) = ch(\mathcal{E}) \wedge \sqrt{td_X}$, and defined the pairing

$$\langle v, w \rangle = \int_X v^\vee \wedge w, \quad v^\vee := \frac{\tau(v)}{\sqrt{ch(\omega_X)}}$$

This vector may also be more modified by setting

$$\mu_\Lambda(\mathcal{E}) := ch(\mathcal{E}) \sqrt{td_X} \cdot \exp(i\Lambda)$$

where $\tau(\Lambda) = -\Lambda$. Thus the former Mukai vector is the special case $\Lambda = 0$. Then

$$\mu_\Lambda(\mathcal{E})^\vee = ch(\mathcal{E})^\vee \cdot \sqrt{td_X} \cdot \exp(-i\Lambda)$$

Knowing $\tau(td_X) = ch(\omega_X)td_X$, and we would still have

$$\langle \mu_\Lambda(\mathcal{E}), \mu_\Lambda(\mathcal{F}) \rangle = \int \mu_\Lambda(\mathcal{E})^\vee \mu_\Lambda(\mathcal{F}) = \int ch(\mathcal{E})^\vee ch(\mathcal{F}) \cdot td_X$$

In this way the replacement for the square root of td_X is a multiplicative characteristic class, namely complex Gamma class, [D], [HJLM],

$$\hat{\Gamma}_X^{\mathbb{C}} = \sqrt{td_X} \exp(i\Lambda_X)$$

By the machinery introduced in the former Sections we may easily study the positivity of Serre multiplicity over \mathbb{C} . First we write the Serre-Cartan-Eilenberg Euler characteristic as,

$$(14) \quad \chi(\mathcal{E}, \mathcal{F}) = \int_X \mu(\mathcal{E})^\vee \wedge \mu(\mathcal{F}),$$

In this way we need to study the positivity of the right hand side using Riemann-Hodge bilinear relations for pure Hodge structures on $H^*(X, \mathbb{C})$.

Remark 2.2. *The Mukai pairing $\langle v, w \rangle = \int_X v^\vee \wedge w$ appears in the context of Mirror Symmetry as a mirror to polarization form of a PVHS. This means it is a polarization of the mirror manifold or the PVHS we already have.*

The positivity of intersection multiplicity proved in [S], shows when Y and Z are projective sub-varieties of X , then

$$(15) \quad \int_X \mu(\mathcal{O}_Y)^\vee \wedge \mu(\mathcal{O}_Z) \geq 0,$$

The aforementioned procedure, i.e expressing the Cartan-Eilenberg Euler characteristic as Hermitian type cup product suggests the idea to prove positivity in the serre multiplicity conjecture by Hodge theory. Formulas (30) and (31) directly relate the intersection multiplicity of algebraic cycles in the chow ring to cup product in homology, and thus to polarization of Hodge structures. Thus it provides a way to determine the positivity on both sides.

3. MULTIPLICITY QUESTION OVER ARBITRARY FIELD

We explain two ideas regarding positivity, related to Hodge theory.

(1) Let X be a smooth projective variety / k of dimension d , and L an ample divisor class. L acts on etale cohomology of X and by hard Lefschetz,

$$(16) \quad L^j : H^{n-j}(X(\bar{k}), \mathbb{Q}_l) \cong H^{n+j}(X(\bar{k}), \mathbb{Q}_l)$$

which implies

$$(17) \quad H^{n-j}(X(\bar{k}), \mathbb{Q}_l) = \oplus_k L^k H^{j-2k}(X(\bar{k}), \mathbb{Q}_l)_{prim}$$

that induces a morphism

$$(18) \quad \Lambda : H^j(X(\bar{k}), \mathbb{Q}_l) \rightarrow H^{j-2}(X(\bar{k}), \mathbb{Q}_l(-1))$$

such that for $m \in H^j(X(\bar{k}), \mathbb{Q}_l)^{prim}$, $\Lambda(L^k m) = L^{k-1}.m$, if $k > 0$ and 0 otherwise. The standard conjecture B asserts that Λ is defined algebraically as the action of a correspondence. If $A^j(X)$ is the co-image of the cycle map

$$(19) \quad cl : CH^j(X)_\mathbb{Q} \rightarrow H^{2j}(X(\bar{k}), \mathbb{Q}_l(j))$$

The standard conjecture A asserts that the morphisms

$$(20) \quad L^{n-2j} : A^i(X) \cong A^{n-i}(X), \quad i < n/2,$$

are isomorphisms. This follows from Conjecture B, that says the Lefschetz decomposition is compatible with A^j 's,

$$(21) \quad A^j(X) = \oplus_k L^k . A^{j-k}(X)^{prim}$$

The conjecture I asserts that the pairing,

$$(22) \quad (-1)^j \langle L^{n-2j} a, b \rangle, \quad a, b \in A^j(X)^{prim}$$

is positive definite for $j \leq n/2$. If we assume I, then A would be equivalent to D stating the equivalence of numerical and homological equivalence for cycles on X , [SA]. By the Lefschetz decomposition we have an isomorphism,

$$* : H^{n+j}(X_{\bar{k}}, \mathbb{Q}_l) \rightarrow H^{n-j}(X_{\bar{k}}, \mathbb{Q}_l)$$

such that for $m \in H^i(X_{\bar{k}}, \mathbb{Q}_l)^{prim}$, we have

$$*(L^k m) = (-1)^{i(i+1)/2} L^{n-i-k} m$$

Combined with Poincare duality this defines a pairing on $H^*(X, \mathbb{Q}_l)$ defined by $(m, *n)$. For a non-zero correspondence $\lambda \in A^n(X \times_k X) \subset \text{End}(H^*(X, \mathbb{Q}_l))$ we consider the transpose λ' w.r.t this pairing. Then, if the standard conjecture B and I are satisfied,

$$\chi = \text{Tr}(\lambda' \circ \lambda) > 0$$

The action of the correspondences always determine the pairing on the homologies, by composing the action of diagonal Δ with the product structure. In this way the positivity of the above trace always implies positivity of the Mukai pairing and therefore, the intersection multiplicity. This shows: *If the Grothendieck Standard conjecture I are satisfied then the intersection multiplicity $\chi(M, N)$ is strictly positive on proper intersections for projective varieties /k, [SA].*

(2) The construction of the higher residue pairing originally belonged to K. Saito, [SA1], may be re-phrased in terms of an identification of twisted de Rham complex and the formal complex of poly-vector fields, [LLS]. There are essentially two type of proof for higher residue pairing over \mathbb{C} . The one cited in [SA1] is mainly a comparison of two construction. One an application of local (Serre) duality theorem to Brieskorn

lattices and their duals. This amounts to define the Brieskorn modules $(\mathcal{H}_f^{(-k)}, \nabla : \mathcal{H}^{(-k-1)} \rightarrow \mathcal{H}_f^{(-k)})$ together with their duals $(\check{\mathcal{H}}^{(k)}, \check{\nabla} : \check{\mathcal{H}}^{(k)} \rightarrow \check{\mathcal{H}}^{(k+1)})$ which satisfy a local duality as

$$\mathcal{H}^{(k)} \times \check{\mathcal{H}}^{(k)} \rightarrow \mathcal{O}_S$$

Then this duality is related to the twisted de Rham complex by

$$\hat{\alpha}_k : \widehat{\mathcal{H}^{(-k)}} \cong R^{n+1} f_*(F^{-k}\Omega, \hat{d}), \quad k \geq 1.$$

where F is the Hodge filtration, [?]. Specifically

$$(23) \quad \hat{\alpha} : \widehat{\mathcal{H}^{(0)}} \cong R^{n+1} f_*(F^0\Omega, \hat{d}), \quad k \geq 1.$$

The second method is a duality isomorphism between the twisted de Rham complex and the twisted differential complex of poly-vector fields. This allows to formulate higher residue by the trace map, via symplectic pairing;

$$K^f(,) : \mathcal{H}_{(0)}^f \times \mathcal{H}_{(0)}^f \rightarrow \mathcal{O}_{S,0}[[t]]$$

Both of these constructions are algebraic and can be stated similarly over any field of characteristic 0 and can be applied over Witt ring construction. Thus a proof follows from the formality (algebraicity) of the construction in [SA1] in characteristic 0.

A Witt ring over a ring A or the ring of Witt vectors of A , is a copy of the infinite product A^∞ , with specific sum and products given in each component by polynomials is $\text{char} = p$. Such a ring has characteristic 0. The higher residue pairing is defined over a complete local ring of un-equal characteristic, where the residue field is perfect of $\text{char} > 0$, as:

$$WK^f(,) : W\widehat{\mathcal{H}}_{(0)}^f \times W\widehat{\mathcal{H}}_{(0)}^f \rightarrow W\widehat{\mathcal{O}}_{S,0}[[t]]$$

such that the induced pairing on the Jacobi ring of f over the Witt ring is the Grothendieck pairing. Because the characteristic is 0, the isomorphism mentioned proceeds word by word to in this case and we still get a mirror type identification between these two formal complexes. Then analogous isomorphisms

$$(WPV_S(X)((t)), Q_f = \bar{\partial}_f + t\partial) \rightleftharpoons (WA_S(X)((t)), d + t^{-1}df \wedge \bullet)$$

$$\iota : (WPV_{S,c}(X)[[t]], Q_f) \hookrightarrow (WPV_S(X)[[t]], Q_f)$$

still hold for W_n s and also in the limit for $W(k)$, and we can define,

$$W\mathcal{H}_{(0)}^f := H^*(WPV(X)[[t]], Q_f)$$

By the same method as before we obtain:

$$W\widehat{Res}^f = \sum_k W\widehat{Res}_k^f(\bullet)t^k$$

with $\widehat{Res}_{k,N}^f$ the higher residues. Similarly, we obtain the higher residue pairing

$$WK^f(,) : W\mathcal{H}_{(0)}^f \times W\mathcal{H}_{(0)}^f \rightarrow \mathcal{O}_{S,0}[[t]], \quad WK^f(, 1) := W\widehat{Res}^f$$

Now by applying the completion, we obtain

$$WK^f(,) : W\widehat{\mathcal{H}}_{(0)}^f \times W\widehat{\mathcal{H}}_{(0)}^f \rightarrow W\widehat{\mathcal{O}}_{S,0}[[t]]$$

Theorem 3.1. *(Higher residue pairing on crystalline site) There exists a $K = \text{Frac}(W(k))$ -sesquilinear form*

$$WK^f(,) : W\widehat{\mathcal{H}}_{(0)}^f \times W\widehat{\mathcal{H}}_{(0)}^f \rightarrow W\widehat{\mathcal{O}}_{S,0}[[t]]$$

Let s_1, s_2 be local sections of $W\mathcal{H}_{(0)}^f$, then;

- $WK^f(s_1, s_2) = \overline{WK^f(s_2, s_1)}$.
- $WK^f(v(t)s_1, s_2) = WK^f(s_1, v(-t)s_2) = v(t)WK^f(s_1, s_2)$, $v(t) \in \mathcal{O}_S[[t]]$.
- $\partial_V.WK^f(s_1, s_2) = WK^f(\partial_V s_1, s_2) + WK^f(s_1, \partial_V s_2)$, for any local section of T_S .
- $(t\partial_t + n)WK^f(s_1, s_2) = WK^f(t\partial_t.s_2, s_1) + WK^f(s_1, t\partial_t.s_2)$
- The induced pairing on

$$W\mathcal{H}_{(0)}^f/t.W\mathcal{H}_{(0)}^f \otimes W\mathcal{H}_{(0)}^f/t.W\mathcal{H}_{(0)}^f \rightarrow \bar{K}$$

is the classical Grothendieck residue.

The conjugation is formally done by $\overline{g(t) \otimes \eta} = g(-t).\eta$, for $g \in W(\mathcal{O}_S)$, $\eta \in WA_S(X)$.

The corollary is just the special case $W(\mathbb{F}_p) = \mathbb{Z}_p$. The notion of opposite filtration and formal primitive elements, and good sections may also be generalized to this case easily. By the period isomorphism one can state similar formulas as in 3.1 for etale cohomology. This isomorphism roughly states that crystalline and etal cohomologies each one determines the other one or they are the same in this sense. The period isomorphism is the natural filtered quasi-isomorphism,

$$R\Gamma_{dR}^{alg}(X) \otimes_{\bar{K}} \mathcal{B}_{dr} \rightarrow R\Gamma_{et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{B}_{dR}$$

where K is the field of fractions of $W(k)$ and \mathcal{B}_{dR} is a discrete valuation field whose valuation ring is called Fontaine ring and its residue field is \mathbb{C}_p . It descends to ,

$$R\Gamma_{dR}^{alg}(X) \otimes_{\bar{K}} \mathbb{C}_p \rightarrow R\Gamma_{et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$$

The comparison theorem indicates that for any DVR namely V , there exists a ring $B(V)$, such that for X smooth and proper V -scheme, the etale cohomology of the generic fiber $X/W(k)$ is related to the crystalline cohomology of $X/W(k)$ by

$$H_{et}^*(X \otimes_{W(k)} \bar{K}) \otimes_{\mathbb{Q}_p} B(V) = H_{crys}^*(X/W(k)) \otimes_{W(k)} B(V)$$

with $K = \text{quot } W(k)$ a totally ramified extension of degree e , [FA]. In fact, for $n, i \in \mathbb{N}$, the specialization map induces isomorphisms compatible with the action of Galois group G_K :

$$H^i((X \times_{\mathcal{O}_K} \bar{k})_{et}, \mathbb{Z}/l^n \mathbb{Z}) \cong H^i((X \times_{\mathcal{O}_K} \bar{K})_{et}, \mathbb{Z}/l^n \mathbb{Z})$$

The period isomorphism say that crystalline and etale cohomologies in some way determine one another. Using the period isomorphism we can state the Higher residue pairing on the etale site if the ground field would be \mathbb{C}_p . Simply in theorem 3.1 if we tensor every thing with \mathbb{C}_p we obtain the same result on the etale site over \mathbb{C}_p .

Theorem 3.2. *(Higher residue pairing on etale site) There exists a sesqui-linear form*

$$K_p^f(,) : \hat{\mathcal{H}}_{(0),p}^f \times \hat{\mathcal{H}}_{(0),p}^f \rightarrow \hat{\mathcal{O}}_{S,0}[[t]]$$

Let s_1, s_2 be local sections of $\mathcal{H}_{(0),p}^{f,\mathbb{C}_p}$.

- $K_{et}^f(s_1, s_2) = \overline{K_{et}^f(s_2, s_1)}$.
- $K_{et}^f(v(t)s_1, s_2) = K_{et}^f(s_1, v(-t)s_2) = v(t)K_{et}^f(s_1, s_2)$, $v(t) \in \mathcal{O}_S[[t]]$.
- $\partial_V.K_{et}^f(s_1, s_2) = K_{et}^f(\partial_V s_1, s_2) + K_{et}^f(s_1, \partial_V s_2)$, for any local section of T_S .
- $(t\partial_t + n)K_{et}^f(s_1, s_2) = K_{et}^f(t\partial_t.s_2, s_1) + K_{et}^f(s_1, t\partial_t.s_2)$
- The induced pairing on

$$\mathcal{H}_{(0),p}^f/t.\mathcal{H}_{(0),p}^f \otimes \mathcal{H}_{(0),p}^f/t.\mathcal{H}_{(0),p}^f \rightarrow \mathbb{C}_p$$

is the classical Grothendieck residue.

4. APPENDIX: GROTHENDIECK STANDARD CONJECTURES

We list the Grothendieck Standard conjectures, [CH], [GR]:

- A : Hard Lefschetz on cycles

$$(24) \quad A(X) : .L^{n-2k} : CH^r(X) \cong CH^{n-r}(X)$$

- B : Lefschetz type Standard Conjecture

$$(25) \quad B(X) : *L : \oplus_{i,r} H^i(X)(r) \rightarrow \oplus_{i,r} H^i(X)(r) \quad \text{is algebraic.}$$

- C : Kunneth type Standard Conjecture

$$(26) \quad C(X) : \pi_X^i : H^\bullet(X) \rightarrow H^i(X) \hookrightarrow H^\bullet(X) \quad \text{is algebraic}$$

- D : Homological and numerical equivalence coincide

$$(27) \quad D(X) : \quad \sim_{hom, \mathbb{Q}} = \sim_{num, \mathbb{Q}}$$

- I : Hodge type Standard conjecture

*$I(X)$: the \mathbb{Q} -valued quadratic form $\alpha \mapsto \langle \alpha, *L(\alpha) \rangle$ on $Z_{hom}^\bullet(X)_{\mathbb{Q}}$ is positive definite.*

REFERENCES

- [CA1] A. Caldararu, S. Willerton, Mukai pairing, categorical approach, arxiv.math/0707.2052v1, 2007
- [CA2] A. Caldararu, Mukai pairing, Hochschild structure, arxiv.math/0308079v2, 2003
- [CA3] A. Caldararu, Mukai transformation, Hochschild Konstant Rosenberg isomorphism, arxiv:math/0308080v3, 2004
- [CHA] C. Chan, An intersection multiplicity in terms of Ext-modules, Proc. Amer. Math. Soc. 130 (2002) 327-336
- [CH] F. Charles, Remarks on the Lefschetz standard conjecture and hyperkahler varieties, IRMAR UMR 6625 du CNRS, Universit de Rennes 1, Campus de Beaulieu, 35042, RENNES CEDEX, FRANCE
- [CST] J. Cuntz, G. Skandalis, B. Tsygan, Cyclic homology in non-commutative geometry. Encyclopedia of mathematical sciences, Operator algebras and non-commutative geometry II, 2001.

- [D] B. Dubrovin, Quantum cohomology and isomonodromy deformation, SISSA, Trieste, power point
- [BEI] beilinson A. , P-adic periods and derived de Rham cohomology, Journal of AMS, Vol 25, 715-738, 2012
- [BEO] P. Berthelot, Ogus A. , Notes on crystalline cohomology, Princeton University Press, 1978
- [FA] Faltings G. , Integral crystalline cohomology over very ramified rings, J. Amer. Math. Soc. 12 (1999), 117-144
- [GS1] H. Gillet, C. Soule, Intersection theory using Adams operations, Invent. math 90, 243-277 (1987)
- [GS2] H. Gillet, C. Soule, arithmetic intersection Theory, Publications Mathematiques de l'Institut des Hautes tudes Scientifiques December 1990, Volume 72, Issue 1, pp 94-174
- [GR] A. Grothendieck, Standard conjectures on algebraic cycles, IHES, France, 1968
- [HJLM] J. Halverson, H. Jockers, J. Lapan, D. Morrison, Perturbative corrections to Kahler moduli spaces, UCSB Math 2013
- [KK] K. Kunemann, Some remarks on the arithmetic Hodge index conjecture, Compositio Mathematica (1995) Volume: 99, Issue: 2, page 109-128
- [LLS] Li C., Li S. , Saito K., Primitive forms via polyvector fields, arXiv:1311.1659v3, 2014.
- [L] J. L. Loday, Cyclic homology, Springer Verlag Vol. 301, 1998,
- [MI] J. Milne, Polarization and Grothendieck Standard conjectures, Ann. of Math, 155 (2002) 599-610
- [RA1] A. Ramadoss, Mukai pairing and integral transform for Hochschild homology, arxiv.org
- [RO1] P. Roberts, The vanishing of intersection multiplicities of perfect complexes, Bull. Amer. Math. Soc. (N.S.) Volume 13, Number 2 (1985), 127-130
- [RO2] P. Roberts, Recent developments on Serre Intersection multiplicity conjectures, Gabber proof of non-negativity conjecture, L'Enseignement Mathematique. IIe Srie 01/1998; 44(3)
- [S] J. P. Serre, Local Algebra, Multiplicities, Lecture notes in mathematics 11, Springer Verlag, Newyork, 1961
- [SA] M. Saito, Monodromy filtration and positivity, RIMS, Kyoto University, arxiv:math/000162v6, 2000
- [W] C. Weibel, The Hodge filtration and cyclic homology, J. K-theory, 12 (1997), 145-164
- [SA1] Saito K. , Period mapping associated to a primitive form, Publications of Research Inst. Math. Sci. , Kyoto Univ., Vol 19, No 3, 1983
- [ST] Stack Project, Crystalline cohomology, Cotangent complex

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